Math 455 Winter 2007 Test #3 Review Questions

In each question below, you'll be given a group G and a subgroup H. For each one, answer the following questions:

a. Is H normal in G?

b. If your answer to (a) is no, then explain why not. (In other words, give an example of an element $a \in G$ such that $aH \neq Ha$.)

c. If your answer to (a) is yes, then find a "familiar group" K from the list below such that $G/H \approx K$.

d. If your answer to (a) is yes, then find a homomorphism $f : G/H \to K$ such that $H = \text{Ker}(f)$.

Familiar groups: $\{e\}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_6$, $\mathbb{Z}_7$, $\mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$,$\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_4$,$\mathbb{Z}_3 \times \mathbb{Z}_5$, S_3, S_4, S_5, S_{22}, A_4, A_5, D_8, D_{10}, D_{12}

#1) $G = \mathbb{Z}_{15}$ and $H = \langle 10 \rangle$

#2) $G = S_5$ and $H = \langle (1, 2, 3, 4, 5) \rangle$

#3) $G = \langle x, y \mid x^8 = y^4 = 1, yx = x^3 y \rangle$ and $H = \{x^i y^j \mid i \in \{0, 1, 2, 3, 4, 5, 6, 7\}, j \in \{0, 2\}\}$.

#4) $G = \mathbb{Z}_{10} \times \mathbb{Z}_{12}$ and $H = \langle (2, 3) \rangle$

#5) $G = S_{22}$ and $H = \{\iota\}$

#6) $G = D_{12}$ and $H = G$.

#7) $G = S_7$ and $H = A_7$

#8) G is given by the Cayley table below and $H = \{p, q\}$

	p	q	r	s	t	u	v	w
p	p	q	r	s	t	u	v	w
q	q	p	s	r	u	t	w	v
r	r	s	q	p	v	w	u	t
s	s	r	p	q	w	v	t	u
t	t	u	w	v	q	p	r	s
u	u	t	v	w	p	q	s	r
v	v	w	t	u	s	r	q	p
w	w	v	u	t	r	s	p	q

#9) G is given by the Cayley table above and $H = \{p, q, r, s\}$

#10) $G = D_8 = \langle x, y \mid x^2 = y^6 = 1, yx = x^5 y \rangle$ and $H = \{1, x\}$

Math 455 Winter 2007 Answers to Test #3 Review Questions

#1) $G = \mathbb{Z}_{15}$ and $H = \langle 10 \rangle$

a. Yes, H is normal in G. After all, G is abelian, and any subgroup of an abelian group is normal.

c. Well, $H = \{0, 5, 10\}$. So $|H| = 3$. So $G/H = [G : H] = |G|/|H| = 15/3 = 5$.

Now, 5 is prime, so by the Corollary to Lagrange's Theorem, G/H must be a cyclic group of order 5. That is, $G/H \approx \mathbb{Z}_5$.

d. The things we want to "throw away" are 0, 5, and 10.

Define $f : G \to \mathbb{Z}_5$ by $f(n) = n$.

The elements of G/H are:

$0 + H = \{0, 5, 10\}$

$1 + H = \{1, 6, 11\}$

$2 + H = \{2, 7, 12\}$

$3 + H = \{3, 8, 13\}$

$4 + H = \{4, 9, 14\}$

Define $f : G \to \mathbb{Z}_5$ by:

$f(0) = 0$	$f(5) = 0$	$f(10) = 0$
$f(1) = 1$	$f(6) = 1$	$f(11) = 1$
$f(2) = 2$	$f(7) = 2$	$f(12) = 2$
$f(3) = 3$	$f(8) = 3$	$f(13) = 3$
$f(4) = 4$	$f(9) = 4$	$f(14) = 4$

#2) $G = S_5$ and $H = \langle (1, 2, 3, 4, 5) \rangle$

a. No, H is not normal in G.

b. Let $a = (1, 2)$.

$H = \{\iota, (1, 2, 3, 4, 5), (1, 3, 5, 2, 4), (1, 4, 2, 5, 3), (1, 5, 4, 3, 2)\}$

Then $aH = \{(1, 2), (2, 3, 4, 5), (1, 3, 5)(2, 4), (1, 4)(2, 5, 3), (1, 5, 4, 3)\}$.

But $Ha = \{(1, 2), (1, 3, 4, 5), ...\}$

So $aH \neq Ha$.

How did I find $(1, 2)$? I experimented!

#3) $G = \langle x, y \mid x^8 = y^4 = 1, yx = x^3y \rangle$ and $H = \{x^i y^j \mid i \in \{0, 1, 2, 3, 4, 5, 6, 7\}, j \in \{0, 2\}\}$.

a. Yes, H is normal in G. Here's how I know:

$H = \{1, x, x^2, x^3, x^4, x^5, x^6, x^7, y^2, xy^2, x^2y^2, x^3y^2, x^4y^2, x^5y^2, x^6y^2, x^7y^2\}$

$|H| = 16$

Also note that $|G| = 32$.

So $[G : H] = 32/16 = 2$.

Any subgroup of index 2 is normal. (Think about why!)

c. The order of G/H is $[G : H] = 2$.

Since 2 is prime, we know $G/H \approx \mathbb{Z}_2$.

d. The elements we want to "throw away" are those where the power of y is even.

So define $f : G \to \mathbb{Z}_2$ by $f(x^i y^j) = j$.

Check that f is well-defined.

You may find it useful to list out all the elements of G, organized into cosets of H, and where f maps them, as in #1.

#4) $G = \mathbb{Z}_{10} \times \mathbb{Z}_{12}$ and $H = \langle (2, 3) \rangle$

a. Yes, H is normal in G, since G is abelian.

c. Note that $|H| = 5 \cdot 4 = 20$ and $|G| = 10 \cdot 12 = 120$.

So the order of G/H is $120/20 = 6$.

Also, G/H is abelian, since G is abelian.

There is only one abelian group of order 6, and it is $\mathbb{Z}_2 \times \mathbb{Z}_3 \approx \mathbb{Z}_6$.

d. We have:

$H = \{(0, 0), (2, 3), (4, 6), (6, 9), (8, 0),$

$(0, 3), (2, 6), (4, 9), (6, 0), (8, 3),$

$(0, 6), (2, 9), (4, 0), (6, 3), (8, 6),$

$(0, 9), (2, 0), (4, 3), (6, 6), (8, 9)\}$

It looks like H consists of all elements of the form (a, b), where a is even and b is divisible by 3.

These are the elements we want to "throw away."

So define $f : G \to \mathbb{Z}_2 \times \mathbb{Z}_3$ by $f(a, b) = (a, b)$.

Check that f is well-defined.

You may find it useful to list out all the elements of G, organized into cosets of H, and where f maps them, as in #1.

#5) $G = S_{22}$ and $H = \{\iota\}$

a. Yes, H is normal in G, since $aH = \{a\} = Ha$ for all $a \in G$. (The trivial subgroup is always normal.)

c. $G/H \approx S_{22}$

d. Define $f : G \to S_{22}$ by $f(a) = a$.

#6) $G = D_{12}$ and $H = G$

a. Yes, H is normal in G, since $aH = G = Ha$ for all $a \in G$. (The improper subgroup is always normal.)

c. $\{e\}$

d. Define $f : G \to \{e\}$ by $f(a) = e$.

#7) $G = S_7$ and $H = A_7$

a. Yes, H is normal in G—index 2.

c. $\mathbb{Z}_2$

d. Sign homomorphism

#8) G is given by the given Cayley table and $H = \{p, q\}$

a. Yes. I don't know any clever trick for this one—you just have to check that $aH = Ha$ for all $a \in G$

c. $\mathbb{Z}_2 \times \mathbb{Z}_2$

How did I get this? Look what happens when you "throw away" p and q. Organize the Cayley table into cosets of H. (Maybe use colors, like we did in class.)

The cosets are:

$pH = \{p, q\}$ (Color this one white.)

$rH = \{r, s\}$ (Color this one red.)

$tH = \{t, u\}$ (Color this one blue.)

$vH = \{v, w\}$ (Color this one green.)

The Cayley table for the factor group G/H becomes:

	white	*red*	*blue*	*green*
white	*white*	*red*	*blue*	*green*
red	*red*	*white*	*green*	*blue*
blue	*blue*	*green*	*white*	*red*
green	*green*	*blue*	*red*	*white*

In the factor group, every element has order 2.

d. $H = \{p, q\}$. These are the elements we want to "throw away."

Define $f : G \to \mathbb{Z}_4$ by

$f(p) = f(q) = (0, 0)$

$f(r) = f(s) = (0, 1)$

$f(t) = f(u) = (1, 1)$

$f(v) = f(w) = (1, 0)$

#9) G is given by the Cayley table above and $H = \{p, q, r, s\}$

a. Yes—index 2.

c. $\mathbb{Z}_2$

d. I'll let you try this one.

#10) a. No.   b. $yH \neq Hy$